ENG

A

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Reference

1) Chapter 3

(a) Power Series expansion of functions

(i) $\sin^2 x$ (ii) $\cos x$ (iii) $\sin^{-1} x$, (iv) $\sec^3 x$

(b) Maclaurins theorems

(c) Taylors theorem

2) Chapter 1

(a) Eulers formula – [$e^{j\theta} = \cos\theta + j\sin\theta$]

(b) Power Series – [$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$]

(c) Power Series of Euler – [$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$]

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

Note:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Thus:

$$\cos x = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}$$

$$\sin x = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

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$e^{j\theta} = \cos\theta + j\sin\theta$

(i) Evaluate $e^{j\frac{3\pi}{2}}$ $\boxed{+ e^{j\frac{\pi}{2}}}$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{j\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + j\sin\left(\frac{3\pi}{2}\right)$

$= \cos 270^\circ + j\sin 270^\circ$

$= 0 + j(-1)$

$= -j$

(ii) Evaluate $e^{j\theta}$ for values of θ: $\frac{\pi}{4}$, $\pi/6$

(iii) Evaluate $e^{j\frac{\pi}{3}} e^{jn2\pi}$

c

Homogenous linear differential equations with constant coefficients can be solved by algebraic methods. However the solution of differential equations with variable coefficient are usually in the form of a power series.

If a function can be represented by a power series of x, that series must be of the form of Maclaurins series, thus we have

$$F(x) = F(0) + \frac{F'(0)x}{1!} + \frac{F''(0)x^2}{2!} + \frac{F'''(0)x^3}{3!} + \cdots + \frac{F^{(n-1)}x^{(n-1)}}{(n-1)!} + \cdots$$

Similarly, if a given function can be represented by a power series in $(x-m)$, that series must be of the form of Taylor series, thus we have

$$F(x) = F(m) + \frac{F'(m)(x-m)}{1!} + \frac{F''(m)(x-m)^2}{2!} + \frac{F'''(m)(x-m)^3}{3!} + \cdots$$

Thus, Differential equations with variable coeffi-
-cients solutions can be shown to be non-ele-
-mentary functions.

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} + \cdots$$

$$\cdots + \frac{x^{n-1} f^{n-1}}{(n-1)!} + \cdots$$

Proof of Maclaurin Series

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$F(0) = a_0 \Rightarrow a_0 = f(0)$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4$$

$$f'(0) = a_1 \Rightarrow a_1 = f'(0)$$

$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$$

$$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + \cdots$$

$$f'''(0) = 6a_3 \Rightarrow a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

$$f^{iv}(x) = 24a_4 + 120a_5 x + \cdots$$

$$f^{iv}(0) = 24a_4 \Rightarrow a_4 = \frac{f^{iv}(0)}{24} = \frac{f^{iv}(0)}{4!}$$

Thus :

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2!}, \quad a_3 = \frac{f'''(0)}{3!}, \quad a_4 = \frac{f^{iv}(0)}{4!}$$

(E)

Substituting $a_0, a_1, \ldots a_4$ in the series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

we have -

$$F(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!}$$

Maclaurins Series may be used to represent any function, say $f(x)$, as power series provided that at $x = 0$, the following three Conditions are Satisfied -

(a) $f(0) \neq \infty$

If $f(x) = \cos x$

$f(0) = \cos 0 = 1$, Thus $\cos x$ meets the Condition

But If $f(x) = \ln x$

$f(0) = \ln 0 = -\infty$; $\ln x$ do not meet the Condition

or

$f(x) = 1/x$

$f(0) = \frac{1}{0} = \infty$, $1/x$ do not meet the Condition

(b) $f'(0), f''(0), f'''(0), \ldots \neq \infty$

(c) The resultant Maclaurins series must be convergent. This means that the value of the terms or groups of terms, must get progressively

- smaller and the sum of the terms must reach a limiting value.

* Determine Power Series for $\sin x$ using Maclaurin Series

$f(x) = \sin x \Rightarrow f(0) = \sin(0) = 0$

$f'(x) = \cos x = f'(0) = \cos(0) = 1$

$f''(x) = -\sin x = f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

using Maclaurin series $f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Questions

1) Determine the Power Series of $\sin^2 2x$ till the term in x^6

2) $\cos^2 x$ till the term in x^6

CHAPTER 6

Standard Series of Some Common Functions

1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

3) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

4) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

5) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

7) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

8) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

9) $\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \cdots$

Maclaurin Series is the taylors expansion of $f(x)$ at $x = m = 0$. Most Power Series representation of certain elementary functions are from the Maclaurins Series.

H

Find the Series representation of the following

(i) $y = e^{x^2}$ (ii) $y = \cos 3x$ (iii) $y = \sin x^3$

(iv) $y = \tan^{-1} x$

Solution

(i) $y = e^{x^2}$

But $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

replacing x by x^2, we have

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{x^{2n}}{n!}$$

(ii) $y = \cos 3x$.

But $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

replacing x with $3x$, i.e $x \equiv 3x$, we have

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots$$

$$\cos 3x = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \cdots$$

6-2

Power Series Solution of First Order Linear Differential Equations

A Power Series in Powers of $(x-m)$ is an Infinite Series of the form

$$y = \sum_{n=0}^{\infty} a_n (x-m)^n$$

Expanding:

$$y = a_0 + a_1(x-m) + a_2(x-m)^2 + a_3(x-m)^3 + \dots + a_n(x-m)^n + \dots$$

Where $a_1, a_2, a_3 \dots a_n$ are Constants, m is another Constant known as the centre, and x is the Independent Variable. If $m=0$, then the series is centred at the origin, and we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

<u>Example</u>: Solve the differential equation by Power Series method $y' - y = 0$

Solution:

$$\text{Let } y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n \quad \text{--- (1)}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots \quad \text{--- (2)}$$

Putting y and y' in the original equation

$$y' - y = a_1 + 2a_2x + 3a_2x^2 + \cdots - (a_0 + a_1x + a_2x^2 + a_3x^3)$$
$$= 0$$

collecting like terms

$$a_1 - a_0 = 0 \Rightarrow a_1 = a_0$$

$$2a_2x - a_1x = 0 \Rightarrow 2a_2x = a_1x \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$3a_3x^2 - a_2x^2 = 0 \Rightarrow 3a_3x^2 = a_2x^2 \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3\times 2}$$

$$4a_4x^3 - a_3x^3 = 0 \Rightarrow 4a_4x^3 = a_3x^3 \Rightarrow a_4 = \frac{a_3}{4} = \frac{a_0}{4\times 3\times 2}$$

Substituting the values of a_1, a_2, a_3 in eqn (1)

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y = a_0 + a_0x + \frac{a_0x^2}{2!} + \frac{a_0x^3}{3\times 2} + \frac{a_0x^4}{4\times 3\times 2}$$

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

Thus the above expression is recognizable as the exponential series of e^x. we write

$$y = a_0e^x$$

Solve the differential equation $(1+x)y' + xy = 0$

Solution

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$ (1)

$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$ (2)

Putting this in original equation

$(1+x)y' + xy = 0$

$= (1+x)[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots] + x[a_0 +$
$a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots] = 0$

$a_1 + (2a_2 + a_1 + a_0)x + (3a_3 + 2a_2 + a_1)x^2 + (4a_4 + 3a_3$
$+ a_2)x^3 + (5a_5 + 4a_4 + a_3)x^4 + 6a_6 + 5a_5$
$+ a_4)x^5 = 0$

Comparing coefficients of terms of x, we obtain

$a_1 = 0$

$2a_2 + a_1 + a_0 = 0 \rightarrow a_2 = \frac{-a_0}{2}$

$3a_3 + 2a_2 + a_1 = 0 \rightarrow a_3 = \frac{a_0}{3}$

$4a_4 + 3a_3 + a_2 = 0 \rightarrow a_4 = -\frac{a_0}{8}$

$5a_5 + 4a_4 + a_3 = 0 \rightarrow a_5 = \frac{a_0}{30}$

substituting in equation (1)

$y = a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{3}x^3 - \frac{a_0}{8}x^4 + \frac{a_0}{30}x^5$

$$f(x)=f(0)+\frac{xf'(0)}{1!}+\frac{x^2f''(0)}{2!}+\frac{x^3f'''(0)}{3!}+\frac{x^4f^{iv}(0)}{4!}+\cdots$$

$$\cdots+\frac{x^{n-1}f^{n-1}}{(n-1)!}+\cdots$$

Proof of Maclaurin Series

Let $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$

$F(0)=a_0 \Rightarrow a_0=f(0)$

$f'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4$

$f'(0)=a_1 \Rightarrow a_1=f'(0)$

$f''(x)=2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots$

$f''(0)=2a_2 \Rightarrow a_2=\frac{f''(0)}{2!}$

$f'''(x)=6a_3+24a_4x+60a_5x^2+\cdots$

$f'''(0)=6a_3 \Rightarrow a_3=\frac{f'''(0)}{6}=\frac{f'''(0)}{3!}$

$f^{iv}(x)=24a_4+120a_5x+\cdots$

$f^{iv}(0)=24a_4 \Rightarrow a_4=\frac{f^{iv}(0)}{24}=\frac{f^{iv}(0)}{4!}$

Thus :

$a_0=f(0)$, $a_1=f'(0)$, $a_2=\frac{f''(0)}{2!}$, $a_3=\frac{f'''(0)}{3!}$, $a_4=\frac{f^{iv}(0)}{4!}$

Substituting $a_0, a_1, \ldots a_4$ in the series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

we have -

$$F(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} + \frac{x^4f^{iv}(0)}{4!}$$

Maclaurins Series may be used to represent any function, say $f(x)$, as power series provided that at $x = 0$, the following three Conditions are Satisfied -

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* Determine Power Series for $\sin x$ using Maclaurin Series

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$f''(x) = -\sin x = f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

Using Maclaurin Series $f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

<u>Questions</u>

1) Determine the Power Series of $\sin^2 2x$ till the term in x^6

2) $\cos^2 x$ till the term in x^6

(A) (3)

nth Order differential equation

A differential equation of nth order is linear if it is of the form:

$$y^n + a_{n-1}(x)y^{n-1} + a_{n-2}(x)y^{n-2} + \dots a_1(x)y' + a_0(x)y = b(x) \quad (1)$$

where b and the coefficients $a_0, a_1 \dots a_{n-1}$ are given functions of x, and y^n is the nth derivative of y with respect to x.

An nth order differential equation which is not of the form 1 is non linear.

If $b(x) = 0$, the equation (1) becomes

$$y^n + a_{n-1}(x)y^{n-1} + a_{n-2}y^{n-2} + \dots + a_1(x)y' + a_0(x)y = 0 \quad (2)$$

and is said to be homogenous.

If $b(x) \neq 0$ the equation is non homogenous

A solution of an nth order differential equation is called a general solution if it has n arbitrary independent constants. Independence means that the solution cannot be reduced to a form containing less than n arbitrary constants.

(B)

If the n constants assume definite values, the solution becomes a particular solution of the differential equation.

A fundamental system of solutions of equation (2) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Solve

(1) $e^{2x} y' = 8$, find y!

$$y' = 8e^{-2x}$$

$$y = \int 8e^{-2x}\,dx + c$$

$$y = -4e^{-2x} + c$$

(2) $y' = \cos 2x$

$$y = \int \cos 2x + c = \tfrac{1}{2}\sin 2x + c$$

(3) $$y' = \frac{x^3 + 2}{x^2}$$

$$y = \int x + 2x^{-2}\,dx + c$$

$$y = \frac{x^2}{2} + \frac{2x^{-1}}{-1} + c = \frac{x^2}{2} - \frac{2}{x} + c$$

(C)

(4) $y' = xe^{x^2}$

$y' = \frac{dy}{dx} = xe^{x^2}$

$y = \int xe^{x^2}dx + c$

$y = \frac{1}{2}e^{x^2} + c$

(5) $y' = x^2\cos x^3$

$y' = \frac{dy}{dx} = x^2\cos x^3$

$y = \int x^2\cos x^3 dx + c$

$= \frac{1}{3}\sin x^3 + c.$

(6) Solve

$\sin x \cos y \, dx = \sin y \cos x \, dy$

Seperating Variables

$\frac{\sin x \, dx}{\cos x} = \frac{\sin y}{\cos y} dy$

$\tan x \, dx = \tan y \, dy$

$\int \tan x \, dx = \int \tan y \, dy$

$-\log \cos = -\log\cos y + \log c$

$\log c = \log\cos y - \log\cos x$

$\log_c = \log\frac{\cos y}{\cos x}$

$c = \frac{\cos y}{\cos x}$

$\cos y = c\cos x$ or

(7) $\frac{1+x^2}{1+y} = xy\frac{dy}{dx}$

Seperating Variables

$(1+y)y\,dy = \frac{1+x^2}{x}dx$

$= x + \frac{1}{x}dx$

Integrating

$\frac{y^2}{2} + \frac{y^3}{3} = \frac{x^2}{2} + \ln x + c$

$c = \frac{y^2}{2} + \frac{y^3}{3} - \frac{x^2}{2} \ln x$

Solve

8) $xy' = x + y$

$$y' = \frac{x+y}{x} = 1 + \frac{y}{x}$$

let $\frac{y}{x} = u(x)$

then $y' = 1 + u$

from product rule

$$y' = u + xu' = 1 + u$$

$$xu' = x\frac{du}{dx} = 1$$

seperating variables we have

$$du = \frac{dx}{x}$$

Integrating

$$\int du = \int \frac{dx}{x}$$

$$u = \ln x + c = \frac{y}{x}$$

$$y = x(\ln x + c)$$

9) $xy' = e^{-xy} - y$

$$y' = \frac{e^{-xy} - y}{x} = \frac{e^{-xy}}{x} - \frac{y}{x}$$

let $u(x) = xy$, then $y = \frac{u(x)}{x}$

from quotient rule

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \left(\frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}\right)$$

Where $u(x)$, and $v(x)$

We have

$$y' = \frac{xv'(x) - v}{x^2}$$

Differential equation becomes:

$$\frac{xv' - v}{x^2} = \frac{e^{-v}}{x} = \frac{v}{x^2}$$

$$xv' - v = xe^{-v} - v$$

$$xe^{-v} = xv' - v' = e^{-v} = \frac{dv}{dx}$$

Seperating variables

$$\frac{dv}{e^{-v}} = dx = e^{v}dv$$

Integrating

$$\int e^{v}dv = \int dx; \quad e^{v} = x + c = e^{xy}$$

Taking logs to base e

$$\ln e^{xy} = \ln(x+c) = xy$$

$$\therefore y = x^{-1}\ln(x+c)$$

(E)

Second Order Equation Reducible to the first order

Consider a second order differential equation which does not contain $y(x)$ explicitly, the second order differential equation can be written in the form:

$F(x, y', y'') = 0$, if we let $y' = u(x)$

then $y'' = u'(x)$; and second order equation becomes:

$f[x, u(x), u'(x)] = 0$

which is of first order in the variables, $u(x)$ and its derivative $u'(x)$. Solution of the first order is obtained with any of the methods while second order is by integration

(9) Reduce to first order

Solve

$2xy'' = 3y'$

Let $u(x) = y'$; $u'(x) = y''$

then $2xu' = 3u$

Seperating Variables

$\frac{u'}{u} = \frac{3}{2}x$

$\frac{du}{u\,dx} = \frac{3}{2}x$

$\frac{du}{u} = 1.5\frac{dx}{x}$

Integrating; $\int\frac{du}{u} = \int 1.5\frac{dx}{x} = 1.5\int\frac{dx}{x}$

$\ln u = 1.5\ln x + \ln c_1$

$\ln u = \ln x^{1.5} + \ln c_1$

$\ln c_1 = \ln u - \ln x^{1.5}$

$\ln c_1 = \ln\left(\frac{u}{x^{1.5}}\right)$

$\therefore c_1 = \frac{u}{x^{1.5}}$; $u = c_1 x^{1.5}$

But $u = y' = c_1 x^{1.5}$

$\frac{dy}{dx} = c_1 x^{1.5}$

$dy = c_1 x^{1.5} dx$

Integrating

(9)

$$\int dy = \int c_1 x^{1.5} dx + c_2$$

$$y = \frac{c_1 x^{2.5}}{2.5} + c_2$$

$$y = 0.4 c_1 x^{2.5} + c_2$$

(10)

$$xy'' + 2y' = 0$$

let $u = y'$, then $u' = y''$

and $xu' + 2u = 0$; $x\frac{du}{dx} = -2u$

Seperating Variables

$$\frac{du}{u} = -2\frac{dx}{x}$$

Integrating

$$\int \frac{du}{u} = -2\int \frac{dx}{x}$$

$$\ln u = -2\ln x + \ln c$$

$$\ln u = \ln x^{-2} + \ln c$$

$$\ln c = \ln u + 2\ln x$$

$$\ln c = \ln u + \ln x^2$$

$$\ln c = \ln (ux^2)$$

$$\therefore c = ux^2 ; \quad u = \frac{c}{x^2} = cx^{-2}$$

But

$$u = y' = cx^{-2}$$

$$\frac{dy}{dx} = cx^{-2}$$

$$dy = cx^{-2} dx$$

Integrating

$$\int dy = y = \int cx^{-2} dx + c_2$$

$$y = \frac{cx^{-1}}{-1} + c_2$$

$$y = -\frac{c}{x} + c_2$$

$$y = -cx^{-1} + c_2$$

Higher Order Derivatives

generally

$$a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y' + a_0 y = b(x) \quad (1)$$

If $b(x) = 0$, the equation is Homogenous else its non-Homogenous. For a Complementary solution of the form $y = e^{mx}$

$$y' = me^{mx} = my, \quad y'' = m^2 e^{mx} = m^2 y$$

$$y''' = m^3 e^{mx} = m^3 y, \quad y^n = m^n y$$

If $y_h = e^{mx}$ is a Complementary solution, then -

$$a_n y_h^n + a_{n-1} y_h^{n-1} + \dots + a_1 y_h' + a_0 y_h = 0 \quad (2)$$

then

$$a_n m^n y + a_{n-1} m^{n-1} y + \dots + a_1 m y + a_0 y = 0$$

or

$$y(a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0) = 0$$

For non trivial solutions

$$y = e^{mx} \neq 0$$

For nth order polynomial in m; we have

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0 \quad \dots \quad (3)$$

The nth order polynomial in equation 3 is called the auxilliary equation, or characteristic equation $\therefore$ three types of roots are possible.

(i) real and distinct root (ii) Multiple root (iii) complex conjugate roots.

Real and distinct roots

When the roots of the auxilliary or characteristics equation are real, ~~and~~ distinct, and given by

$m_1, m_2, m_3 \ldots m_n$

then the basis of linearity independent solutions of the nth order differential equations

$$y_1 = e^{m_1 x},\ y_2 = e^{m_2 x} \text{ is } \ldots y_n = e^{m_n x}$$

The complementary solution is by super position (lin-earity) principle.

$$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Multiple roots

The auxilliary polynomial in 'm' will have a multiplicity n if it can be written as: $(m - m_1)^n = 0$

Generally, if a root $m = m_1$ occurs with a multiplicity k, then the basis of linearity independent solution is

$$e^{m_1 x},\ x e^{m_1 x},\ x^2 e^{m_1 x}, \ldots, x^{k-1} e^{m_1 x}$$

From the Super position (Linearity) principle, the Complementary Solution become:

for multiple roots, m_1 of multiplicity

$$y_h = (c_1 + c_2x + c_3x^2 + c_4x^3 + \dots + c_kx^{k-1})e^{m_1x}$$

By extension for multiple roots m_1 and m_2 of multiplicities k and r respectively, the basis for linearly Independent Solutions become.

$$e^{m_1x}, xe^{m_1x}, x^2e^{m_1x}, \dots, x^{k-1}e^{m_1x}$$

$$e^{m_2x}, xe^{m_2x}, x^2e^{m_2x}, \dots x^{r-1}e^{m_2x}$$

the complementary Solution becomes,

$$y_h = c_0e^{m_1x} + c_1xe^{m_1x} + c_2x^2e^{m_2x} + \dots + c_kx^{k-1}e^{m_1x}$$

$$+ f_0e^{m_2x} + f_1xe^{m_2x} + f_2x^2e^{m_2x} + \dots + f_rx^{r-1}e^{m_2x}$$

$$y_h = (c_0 + c_1 + c_2x^2 + \dots + c_kx^{k-1})e^{m_1x} +$$

$$Cf_0 + f_1x + f_2x^2 + \dots + f_rx^{r-1})e^{m_2x}$$

<u>Solve</u>

(1) $y''' + 4y'' - y' - 4y = 0$

let $y = e^{mx}$ be the Solution, then the auxiliary equation is $m^3 + 4m^2 - m - 4 = 0$

1

factorizing

$(m+1)(m-1)(m+4)=0$

this gives three real and distinct root, thus the basis of solution becomes

$y_1 = e^{-x}, \; y_2 = e^{x}, \; y_3 = e^{-4x}$

the general solution is, by superposition

$y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-4x}$

2 $y^{iv} + 12y''' + 36y'' = 0$

let $y = e^{mx}$ be the solution, then the auxiliary equation is

$m^4 - 12m^3 + 36m^2 = 0$ or $m^2(m+6)^2 = 0$

$m = 0$ (twice), $m = -6$ (twice)

the general solution is

$y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^{-6x}$

$c_1, c_2, c_3,$ and c_4 = Integration constant

(K)

6.3 Higher Order Derivatives

(i) If $y = \sin ax$, then

$$y' = a\cos ax = a\sin\left(ax + \frac{\pi}{2}\right)$$

$$y'' = -a^2\sin ax = a^2\sin(ax+\pi) = a^2\sin\left(ax + \frac{2\pi}{2}\right)$$

$$y''' = -a^3\cos ax = a^3\sin\left(ax + \frac{3\pi}{2}\right)$$

$$y^n = a^n\sin\left(ax + \frac{n\pi}{2}\right)$$

Example: find the Value of y^6, given that $y = \sin 4x$

Solution

$$y^6 = 4^6\sin\left(4x + \frac{6\pi}{2}\right)$$

$$= 4^6\sin(4x + 3\pi)$$

$$= 4096(\sin 4x\cos 3\pi + \cos 4x\sin 3\pi)$$

$$y^6 = -4096\sin 4x$$

Example 2: Determine the Value of y^5 if $y = \sin 2x$

$$y^5 = 2^5\sin\left(2x + \frac{5\pi}{2}\right) = 32\left(\sin 2x\cos\frac{5\pi}{2} + \cos 2x\sin\frac{5\pi}{2}\right)$$

$$y^5 = 32\cos 2x$$

(ii) If $y = \cos ax$, then $y' = -a\sin ax = a\cos\left(ax + \frac{\pi}{2}\right)$

$$y'' = -a^2\cos ax = a^2\cos\left(ax + \frac{2\pi}{2}\right)$$

$$y''' = a^3\sin ax = a^3\cos\left(ax + \frac{3\pi}{2}\right)$$

In general, $y^n = a^n\cos\left(ax + \frac{n\pi}{2}\right)$

Example 3

If $y = 2\cos 3x$, determine the Value of y^5

Solution

$$y^5 = 2\times 3^5\cos\left(3x + \frac{5\pi}{2}\right) = 486\left(\cos 3x\cos\frac{5\pi}{2} - \sin 3x\sin\frac{5\pi}{2}\right)$$

$$y^5 = -486\sin 3x$$

(ii) If $y = e^{ax}$, then $y' = ae^{ax}$, $y'' = a^2e^{ax}$

$$y''' = a^3e^{ax}$$

In general, $y^n = a^ne^{ax}$

Example 4

If $y = 6e^{4x}$, determine the Value of y^4

Solution

$$y^4 = 6\times 4^4e^{4x} = 1536e^{4x}$$

$$y^4 = 1536e^{4x}.$$

6.4

Solution of Second Order Differential Equation with Variable Coefficients

Constant coefficient equations like; $ay'' + by' + cy = 0$ — (1) has a, b, and c independent of x. but are constants. Various techniques like analytical methods involving auxiliary equation, algebraic methods, etc are employed in solving this equation. When the coefficients are variables, and dependent on x, further methods are devised.

Consider a second order differential equation with Variable Coefficients

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (2)$$

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

This cannot be solved by the analytic methods, but in the form of an infinite series of powers of x. Methods that will assist in this include:

Cauchy (or Euler) Method

The general solution to the Cauchy equation is the sum of the homogeneous part and the particular integral. The complementary function is obtained by assuming a solution of the form: $y = y^m$. The solution depends on the nature of the root.

When the roots are real and distinct, that is $m_1 \neq m_2$

The general solution is $y = K_1 x^{m_1} + K_2 x^{m_2} + \dots + K_n x^{m_n}$

N

When the roots are equal, that is $M_1 = M_2 = M$

the general solution is

$$y = x^m (K_1 + K_2 \ln x)$$

and when the roots are complete and conjugate, that is

$m = a \pm jb$, the general solution is

$$y = x^a [K_1 \cos(b \ln x) + K_2 \sin(b \ln x)].$$

Example

Solve the differential equation $x^2 y'' - 3xy' + 3y = 0$

Let $y = x^m \Rightarrow y' = mx^{m-1}; \Rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y', and y'' in the original equation we have:

$$x^2\{m(m-1)\}x^{m-2} - 3x(mx^{m-1}) + 3x^m = 0$$

$$x^2\{m(m-1)\}\frac{x^m}{x^2} - 3xm\frac{x^m}{x} + 3x^m = 0$$

$$x^m\{m(m-1) - 3m + 3\} = 0$$

$x^m \neq 0$ and $m(m-1) - 3m + 3 = 0$

$m^2 - 4m + 3 = 0 \Rightarrow m = 3$ or 1 Hence the general solution is

$$y = K_1 x^{m_1} + K_2 x^{m_2} \Rightarrow y = K_1 x^3 + K_2 x^1$$

$\therefore$ $y = K_1 x^3 + K_2 x$ or $y = x(K_1 x^2 + K_2)$

<u>Example 2</u> Solve the Euler equation: $x^2y'' - xy' + y = 0$

Let $y = x^m$; $y' = mx^{m-1}$; $y'' = m(m-1)x^{m-2}$

Substituting the value of y, y', & y'' in the equation

$$x^2\{m(m-1)\}x^{m-2} - x(mx^{m-1}) + x^m = 0$$

$$m^2 - m - m + 1 = 0; \quad m^2 - 2m + 1 = 0; \quad \therefore m = 1 \text{ (twice)}$$

Hence the general solution

$$y = x^m(K_1 + K_2 \ln x) = x(K_1 + K_2 \ln x)$$

<u>Example 3</u> Solve the differential equation: $x^4y'' - 4x^3y' + 6x^2y = 0$

Dividing through by x^2, we obtain the Cauchy equation

$$x^2y'' - 4xy' + 6y = 0$$

Let $y = x^m$, $\rightarrow y' = mx^{m-1}$; $\rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y', and y'' in the Euler equation -

$$x^2\{m(m-1)\}x^{m-2} - 4x(mx^{m-1}) + 6x^m = 0$$

$$m^2 - 5m + 6 = 0, \quad m = 2 \text{ or } 3$$

$$y = K_1x^{m_1} + K_2x^{m_2} = y = K_1x^2 + K_2x^3$$

6.6

Power Series Solution to Second Order O.D.E

Power series solutions to second order ODE with variable coefficients such as $y''+P(x)y'+q(x)y=0$ where P and q are functions of x, are convenient method

<u>Leibnitz theorem</u> - The nth derivative of a product of two functions.

Let $y = uv$ —— (1), where u and v are functions of x, then by product rule

$y' = uv' + vu'$ where $v' = \frac{dv}{dx}$ an $u' = \frac{du}{dx}$ —— (2)

$$y'' = uv'' + v'u' + vu'' + u'v' = u''v + 2u'v' + uv'' \quad (3)$$

$$y''' = u''v' + vu''' + 2u'v'' + 2v'u'' + uv''' + v''u'$$

$$= u'''v + 3u''v' + 3u'v'' + uv''' \quad (4)$$

Further stage of differentiation will give

$$y^{(4)} = u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)} \quad (5)$$

Where $u^{(0)} = u$ and $v^{(0)} = v$, we have

$$y^{(4)} = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)} \quad (6)$$

From equation (1 - 6), we observe that

(i) Superscript of u decreases regularly by 1 from left to right

(ii) " " v Increases " " " " "

(iii) Coefficients of 1, 4, 6, 1 are the normal binomial coefficients